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On the planarity of the k-zero-divisor hypergraphs

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Abstract

Let R be a commutative ring with identity and let $Z(R, k)$ be the set of all k-zero-divisors in R and $k > 2$ an integer. The *k-zero-divisor hypergraph* of R, denoted by $\mathcal{H}_k(R)$, is a hypergraph with vertex set $Z(R, k)$, and for distinct element $x_1, x_2, \ldots, x_k$ in $Z(R, k)$, the set $\{x_1, x_2, \ldots, x_k\}$ is an edge of $\mathcal{H}_k(R)$ if and only if $x_1 x_2 \cdots x_k = 0$ and the product of elements of no $(k-1)$-subset of $\{x_1, x_2, \ldots, x_k\}$ is zero. In this paper, we characterize all finite commutative non-local rings R for which the k-zero-divisor hypergraph is planar.

Keywords: Hypergraph; Zero-divisor graph; Planar hypergraph; Incidence graph

1. Introduction

The study linking commutative ring theory with graph theory has been stated with the concept of the zero-divisor graph of a commutative ring. Let R be a commutative ring and $Z(R)^*$ be the set of all non-zero zero-divisors of R. The zero-divisor graph of R, denoted by $\Gamma(R)$, is the simple graph with $Z(R)^*$ as the vertex set and two distinct vertices x and y are joined by an edge if and only if $xy = 0$. This definition was introduced by Anderson and Livingston in [1] and later was studied extensively in [2–5]. Later Redmond [6] has extended this concept to any arbitrary ring. In view of this, Eslahchi and Rahimi [7] have introduced and investigated a hypergraph called the *k-zero-divisor hypergraph* of a commutative ring. For a commutative ring R and $k \geq 2$ a fixed integer, a nonzero nonunit element x_1 in R is said to be a *$k-$ zero-divisor* in R if there exist $(k-1)$ distinct nonunit elements $x_2, x_3, \ldots, x_k$ in R different from x_1 such that $\prod_{i=1}^{k} x_i = 0$ and the product of any $(k-1)$ elements of $\{x_1, x_2, \ldots, x_k\}$ is nonzero. By $Z(R, k)$ we denote the set of all *$k-$ zero-divisors* of R. The *k-zero-divisor hypergraph* $\mathcal{H}_k(R)$ of R is defined as the hypergraph with the vertex set $Z(R, k)$, and for distinct elements $x_1, x_2, \ldots, x_k$ in $Z(R, k)$, the set $\{x_1, x_2, \ldots, x_k\}$ is an edge of $\mathcal{H}_k(R)$ if and only if $x_1 x_2 \cdots x_k = 0$ and the product of elements of no $(k-1)$-subset of $\{x_1, x_2, \ldots, x_k\}$ is zero. Note that $\mathcal{H}_2(R) = \Gamma(R)$. For basic definitions on rings, one may refer to [8].

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In [3], the following question was asked: for which finite commutative rings R is $\Gamma(R)$ planar? A partial answer was given in [2], but the question remained open for local rings of order 32. In [4], and independently in both [9] and [10], it was shown that no local ring of order 32 has a planar zero divisor graph. Furthermore, Smith [4] gave a complete list of finite rings and this list included the 2 infinite families $\mathbb{Z}_2 \times F$ and $\mathbb{Z}_3 \times F$, where F is any finite field, and 42 other isomorphism classes of finite rings for which $\Gamma(R)$ planar. All these studies concentrate on the planarity of $\Gamma(R) = \mathcal{H}_2(R)$. Hence in this paper, we study about planarity of $\mathcal{H}_k(R)$ for $k \geq 3$. One can refer to [11,12] for genus characterizations of various other graphs from commutative rings.

This paper is organized as follows: in Section 2, we study some basic properties of the k-zero-divisor hypergraph $\mathcal{H}_k(R)$ of a commutative non-local ring R. In Section 3, we characterize all finite commutative non-local rings R for which the k-zero-divisor hypergraph $\mathcal{H}_k(R)$ is planar. Throughout this paper, we assume that R is a finite commutative non-local ring with identity, $Z(R)$ its set of zero-divisors and $R^\times$ its group of units, $\mathbb{F}_q$ the field with q elements, and $R^* = R - \{0\}$. Now let us summarize notations, concepts and results related to the planarity of graphs and hypergraphs which will be needed in the subsequent sections.

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set V and edge set E. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use K_n to denote the complete graph with n vertices. An r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. The girth of G is the length of a shortest cycle in G and is denoted by $gr(G)$. If G has no cycles, we define the girth of G to be infinite. A graph G is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. For basic definitions on graphs, one may refer to [13,14].

A *hypergraph* $\mathcal{H}$ is a pair $(V(\mathcal{H}),\ E(\mathcal{H}))$ of disjoint sets, where $V(\mathcal{H})$ is a non empty finite set whose elements are called vertices and the elements of $E(\mathcal{H})$ are nonempty subsets of $V(\mathcal{H})$ called edges. The hypergraph $\mathcal{H}$ is called k-uniform if every edge e of $\mathcal{H}$ is of size k. A hypergraph is said to be d-regular if each of its vertices is contained in exactly d edges. It is said to be *linear* if any two of its edges share at most one vertex. A hypergraph is said to be a *cycle* if its edges can be cyclically ordered, say as $(e_1, \ldots, e_\ell)$, such that and $e_1 = e_{\ell+1}$, $e_i \cap e_{i+1} \neq \emptyset$ for all $i \in \{1, \ldots, \ell\}$ and $e_i \cap e_j = \emptyset$ whenever $|i - j| > 1$. The length of a cycle is the number of edges in it. The *girth* of a hypergraph is the length of a shortest cycle it contains. A unicyclic hypergraph is a connected hypergraph containing exactly one cycle. The number of edges containing a vertex $v \in V(\mathcal{H})$ is its degree $d_{\mathcal{H}}(v)$. For basic definitions on hypergraphs, one may refer to [15].

The *incidence graph* $\mathcal{I}(\mathcal{H})$ of $\mathcal{H}$ is a bipartite graph with vertex $V(\mathcal{H}) \cup E(\mathcal{H})$ and a vertex $v \in V(\mathcal{H})$ is adjacent to a vertex $u \in E(\mathcal{H})$ if the vertex v is incident with the hyperedge u in $\mathcal{H}$. The genus $g(\mathcal{H})$, of a hypergraph $\mathcal{H}$ is the genus of its incidence graph; i.e. $g(\mathcal{H}) = g(\mathcal{I}(R))$ [16]. For details on the notion of embedding a graph in a surface, see [14]. We need the following theorems.

Theorem 1.1 (*[13, Kuratowski]*)**.** *A graph G is planar if and only if it contains no subdivision of* K_5 *or* $K_{3,3}$.

Theorem 1.2 (*[16, Corollary 2]*)**.** *A hypergraph is planar if and only if its incidence graph is planar.*

2. Basic properties of $\mathcal{H}_k(R)$

In this section, we study some fundamental properties of $\mathcal{H}_k(R)$. Also, we characterize all finite commutative non-local rings R with identity whose k-zero-divisor hypergraph $\mathcal{H}_k(R)$ has girth 3 or ∞. From the definition, we have the following observations.

Remark 2.1. Let $R = F_1 \times \cdots \times F_n$, where each F_i is a field and $3 \leq k \leq n$. If $\mathbb{A}_\ell = \{x = (a_1, \ldots, a_n) \in R :$ exactly ℓ components in the n tuple x are zero$\}$ for $1 \leq \ell \leq n - k + 1$. Then $Z(R, k) = \bigcup_{\ell=1}^{n-k+1} \mathbb{A}_\ell$. Further when $R = \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3$, $K = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ and $S = R_1 \times \cdots \times R_n$, where $(R_i, \mathfrak{m}_i)$ is a local ring and $\mathfrak{m}_i \neq \{0\}$, for any $k = 3, \ldots, n$,

(i) $\mathcal{H}_k(K)$ is a subhypergraph of both $\mathcal{H}_k(R)$ and $\mathcal{H}_k(S)$
(ii) $\mathcal{H}_k(K)$ and $\mathcal{H}_k(R)$ are subhypergraphs of $\mathcal{H}_k(S)$.

Theorem 2.2. *Let $R = F_1 \times \cdots \times F_n$ be a finite commutative non-local ring, where each F_i is a field. For any k, $3 \le k \le n$, $gr(\mathcal{H}_k(R)) = 3$ or ∞.*

Proof. Suppose there exists at least one F_i such that $|F_i| \ge 3$. Without loss of generality, we assume that $|F_1| \ge 3$.

Let $x_1 = (0, 1, 1, 1, \ldots, 1)$, $x_2 = (a, 0, 1, 1, \ldots, 1)$, $x_3 = (a, 1, 0, 1 \ldots, 1), \ldots, x_{k-1} = (a, 1, 1, \ldots, 1, 0, 1)$, $x_k = (a, 1, 1, \ldots, 1, 1, 0)$, $y_1 = (b, 0, 1, 1, \ldots, 1)$, $y_2 = (b, 1, 0, 1, \ldots, 1), \ldots, y_{k-1} = (b, 1, 1, \ldots, 1, 0) \in Z(R, k)$, where $a, b \in F_1^*$. Then $e_1 = \{x_1, \ldots, x_k\}$, $e_2 = \{x_1, y_1, \ldots, y_{k-1}\}$, $e_3 = \{x_1, \ldots, x_{k-1}, y_{k-1}\} \in E(\mathcal{H}_k(R))$ and so $e_1 - e_2 - e_3 - e_1$ is a cycle in $\mathcal{H}_k(R)$. Hence $gr(\mathcal{H}_k(R)) = 3$.

Assume that $|F_i| = 2$ for all i. Then $R \cong \mathbb{Z}_2^n$. If $k = n$, then $|E(\mathcal{H}_n(\mathbb{Z}_2^n))| = 1$ and so $gr(\mathcal{H}_k(R)) = \infty$.

Suppose $3 \le k < n$.

Let $y_1 = (\underbrace{0, 1, \ldots, 1,}_{k\text{-terms}} \overbrace{0, \ldots, 0}^{n-k\ terms})$, $y_2 = (\underbrace{1, 0, 1, \ldots, 1,}_{k\ terms} \overbrace{0, \ldots, 0}^{n-k\ terms}), \ldots,$

$y_k = (\underbrace{1, 1, \ldots, 1, 0,}_{k\ terms} \overbrace{0, \ldots, 0}^{n-k\ terms})$, $y_2' = (\underbrace{1, 0, 1, \ldots, 1,}_{k\ terms} \overbrace{1, 0, \ldots, 0}^{n-k\ terms})$,

$y_3' = (\underbrace{1, 1, 0, 1, \ldots, 1,}_{k\ terms} \overbrace{1, 0, \ldots, 0}^{n-k\ terms}) \in V(\mathcal{H}_k(R))$. Then $e_1 = \{y_1, \ldots, y_k\}$, $e_2 = \{y_1, y_2', y_3, \ldots, y_k\}$, $e_3 = \{y_1, y_2, y_3', y_4, \ldots, y_k\} \in E(\mathcal{H}_k(R))$ and $e_1 - e_2 - e_3 - e_1$ is a cycle in $\mathcal{H}_k(R)$ and hence $gr(\mathcal{H}_k(R)) = 3$. $\square$

In the next theorem, we characterize all finite commutative non-local rings R for which $gr(\mathcal{H}_k(R)) = 3$ or ∞.

Theorem 2.3. *Let $R = R_1 \times \cdots \times R_n$ be a finite commutative non-local ring, where each R_i is a local ring. Then*

(i) *$gr(\mathcal{H}_3(R)) = \infty$ if and only if $R \cong \mathbb{Z}_4 \times \mathbb{Z}_3$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

(ii) *If $3 < k = n$, then $gr(\mathcal{H}_k(R)) = \infty$ if and only if $R \cong \mathbb{Z}_2^n$.*

Proof. (i) Assume that $gr(\mathcal{H}_3(R)) = \infty$. Suppose $n \ge 4$.

Let $x_1 = (0, 1, 1, 0, \ldots, 0)$, $x_2 = (1, 0, 1, 0, \ldots, 0)$, $x_3 = (1, 1, 0, 0, \ldots, 0)$, $x_4 = (1, 0, 1, 1, 0, \ldots, 0)$, $x_5 = (1, 1, 0, 1, 0, \ldots, 0) \in V(\mathcal{H}_3(R))$. Then $e_1 = \{x_1, x_2, x_3\}$, $e_2 = \{x_1, x_4, x_3\}$, $e_3 = \{x_1, x_2, x_5\} \in E(\mathcal{H}_3(R))$ and so $e_1 - e_2 - e_3 - e_1$ is a cycle of length 3 in $\mathcal{H}_3(R)$, a contradiction. Hence $n = 2$ or 3.

Assume that $n = 3$. Suppose there exists at least one ring R_i such that $|R_i| \ge 3$. As noted in the proof of Theorem 2.2, $gr(\mathcal{H}_3(R)) = 3$, a contradiction. Hence $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Assume that $n = 2$. If R_1 and R_2 are fields, then $Z(R, 3) = \emptyset$. Hence at least one of R_i is such that $Z(R_i) \ne \{0\}$. Without loss of generality, we assume that $Z(R_1) \ne \{0\}$. If $|Z(R_1)^*| \ge 2$, then $|R_1^\times| \ge 3$. Let $a, b \in Z(R_1)^*$ be such that $ab = 0$ and $u_1, u_2, u_3 \in R_1^\times$. Let $y_1 = (a, 1)$, $y_2 = (b, 1)$, $y_3 = (u_1, 0)$, $y_4 = (u_2, 0)$, $y_5 = (u_3, 0) \in Z(R, 3)$. Then $e_1 = \{y_1, y_2, y_3\}$, $e_2 = \{y_1, y_2, y_4\}$, $e_3 = \{y_1, y_2, y_5\} \in E(\mathcal{H}_3(R))$ and so $e_1 - e_2 - e_3 - e_1$ is a cycle in $\mathcal{H}_3(R)$, a contradiction. Hence $|Z(R_1)^*| = 1$ and so $R_1 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.

Suppose $|R_2| \ge 4$. Let $z_1 = (a, v_1)$, $z_2 = (a, v_2)$, $z_3 = (a, v_3)$, $z_4 = (u_1, 0)$, $z_5 = (u_2, 0) \in Z(R, 3)$, where $a \in Z(R_1)^*$, $a^2 = 0$, $u_1, u_2 \in R_1^\times$ and $v_1, v_2, v_3 \in R_2^*$. Then $e_1 = \{z_1, z_2, z_4\}$, $e_2 = \{z_1, z_3, z_4\}$, $e_3 = \{z_1, z_3, z_5\} \in E(\mathcal{H}_3(R))$ and so $e_1 - e_2 - e_3 - e_1$ is a cycle in $\mathcal{H}_3(R)$, a contradiction. Hence $|R_2| \le 3$. Since $Z(\mathbb{Z}_4 \times \mathbb{Z}_2, 3) = Z\left(\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2, 3\right) = \emptyset$ and $R_2 \ncong \mathbb{Z}_2$, we arrive that $R_2 \cong \mathbb{Z}_3$. Hence $R \cong \mathbb{Z}_4 \times \mathbb{Z}_3$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_3$.

Converse is obvious.

(ii) Proof follows from Theorem 2.2. $\square$

Theorem 2.4. *Let $R = F_1 \times \cdots \times F_n$, where each F_i is a field and $3 \le k \le n$. Then there exists a vertex $v \in V(\mathcal{H}_k(R))$ such that $v \in e$ for all $e \in E(\mathcal{H}_k(R))$ if and only if $k = n$ and at least $(n - 1)$ fields in the product are $\mathbb{Z}_2$.*

Proof. Assume that there is a vertex $v \in V(\mathcal{H}_k(R))$ such that $v \in e$ for all $e \in E(\mathcal{H}_k(R))$. Suppose $k < n$. As mentioned in Remark 2.1, $Z(R, k) = \bigcup_{\ell=1}^{n-k+1} \mathbb{A}_\ell$ and so $v \in \mathbb{A}_t$ for some t. Let $v = (a_1, \ldots, a_n) \in A_t$, where

$a_{i_s} \neq 0$, $a_{i_j} = 0$ for $s \neq j$, $1 \leq j \leq t$. Let $I = \{1, 2, \dots, n\}$, $I_1 = \{i_j : 1 \leq j \leq t\}$ and $I_2 = I - I_1$. Note that $|I| = n$, $|I_1| = t$ and $|I_2| = n - t$. Suppose $k < t$. Let $I_3 \subset I_1$ with $|I_3| = t - k$. For each $\ell = 1, \dots, k$, $v_\ell = (b_{\ell 1}, \dots, b_{\ell n}) \in V(\mathcal{H}_k(R))$ where $b_{\ell j_s} = 0$ for $\ell j_s \in I_2 \cup I_3 \cup \{i_\ell\}$ and $b_{\ell j_a} = 1$ for $\ell j_a \in I_1 - (I_3 \cup \{i_\ell\})$. Then $e = \{v_1, \dots, v_k\}$ is an edge in $\mathcal{H}_k(R)$ and $v \notin e$, a contradiction.

Suppose $k > t$. Let $I_3 \subset I_2$ be such that $|I_3| = k - t$. For each $\ell = 1, \dots, k$, $u_\ell = (c_{\ell 1}, \dots, c_{\ell n}) \in V(\mathcal{H}_k(R))$ where $c_{\ell j_s} = 0$ for $\ell j_s \in (I_2 - I_3) \cup \{i_\ell\}$ and $c_{\ell j_a} = 1$ for $\ell j_a \in (I_1 \cup I_3) - \{i_\ell\}$. Then $e = \{u_1, \dots, u_k\}$ is an edge in $\mathcal{H}_k(R)$ and $v \notin e$, a contradiction. Hence $k = n$.

Suppose $R_i \ncong \mathbb{Z}_2$ for at least two i. Without loss of generality, we assume that $|R_{n-1}| \geq 3$, $|R_n| \geq 3$ and so $R = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times R_{n-1} \times R_n$.

Consider the partition

$W_1 = \{(0, 1, 1, \dots, 1, a, b) : a \in R^*_{n-1}, b \in R^*_n\}$,

$W_2 = \{(1, 0, 1, \dots, 1, a, b) : a \in R^*_{n-1}, b \in R^*_n\}, \dots$,

$W_{k-1} = \{(1, 1, \dots, 1, 0, b) : b \in R^*_n\}$,

$W_k = \{(1, 1, \dots, 1, a, 0) : a \in R^*_{n-1}\}$ of $V(\mathcal{H}_k(R))$. Thus $V(\mathcal{H}_k(R)) = \bigcup_{i=1}^n W_i$.

For each $i = 1, 2, \dots, k$ and any $x, y \in W_i$, $x, y \notin e$ for all $e \in E(\mathcal{H}_k(R))$, a contradiction. Hence at least $(n - 1)$ fields in the product are $\mathbb{Z}_2$.

Conversely, if $k = n$ and $R \cong \mathbb{Z}_2^n$, then $|E(\mathcal{H}_n(R))| = 1$ and so the proof is trivial. Suppose $R = R_1 \times \mathbb{Z}_2^{n-1}$ and $R_1 \ncong \mathbb{Z}_2$. Then $v = (0, 1, 1, \dots, 1) \in e$ for all $e \in E(\mathcal{H}_k(R))$. $\square$

A hypertree is a connected linear hypergraph without cycles. The next theorem shows that any hypertree cannot be realized as $\mathcal{H}_k(R)$.

Theorem 2.5. *Let $R = R_1 \times \cdots \times R_n$ be finite commutative ring, where each R_i is a local ring and $|E(\mathcal{H}_k(R))| \geq 2$. Then for any k, $3 \leq k \leq n$, $\mathcal{H}_k(R)$ is not linear and hence $\mathcal{H}_k(R)$ is not a hypertree.*

Proof. Suppose $k = n$. Since $|E(\mathcal{H}_k(R))| \geq 2$, $|R_i| \geq 3$ for some i. Without loss of generality, we assume that $|R_1| \geq 3$. Let $a, b \in R_1^*$. Then $x_1 = (0, 1, 1, \dots, 1)$, $x_2 = (a, 0, 1, \dots, 1)$, $x_3 = (a, 1, 0, \dots, 1)$, …, $x_k = (a, 1, 1, \dots, 0)$, $x'_3 = (b, 1, 0, \dots, 1) \in V(\mathcal{H}_k(R))$. From this, we get $e_1 = \{x_1, x_2, x_3, \dots, x_k\}$, $e_2 = \{x_1, x_2, x'_3, x_4 \cdots, x_k\} \in E(\mathcal{H}_k(R))$ and $|e_1 \cap e_2| \geq 2$. Hence $\mathcal{H}_k(R)$ is not linear.

Suppose $k < n$. Let $x_1 = (\underbrace{0, 0, 0, \dots, 0}_{n-k+1\ terms}, \overbrace{1, 1, \dots, 1}^{k-1\ terms})$, $x_2 = (\underbrace{a, 1, 1, \dots, 1}_{n-k+1\ terms}, \overbrace{0, 1, \dots, 1}^{k-1\ terms})$, $x_3 = (\underbrace{a, 1, 1, \dots, 1}_{n-k+1\ terms}, \overbrace{1, 0, 1, \dots, 1}^{k-1\ terms})$, …, $x_{k-1} = (\underbrace{a, 1, 1, \dots, 1}_{n-k+1\ terms}, \overbrace{1, \dots, 1, 0, 1}^{k-1\ terms})$, $x_k = (\underbrace{a, 1, 1, \dots, 1}_{n-k+1\ terms}, \overbrace{1, \dots, 1, 0}^{k-1\ terms})$, $x'_3 = (\underbrace{b, 1, 1, \dots, 1}_{n-k+1\ terms}, \overbrace{1, 0, 1, \dots, 1}^{k-1\ terms}) \in V(\mathcal{H}_k(R))$. Then $e_1 = \{x_1, x_2, x_3, \dots, x_k\}$ and $e_2 = \{x_1, x_2, x'_3, \dots, x_k\} \in E(\mathcal{H}_k(R))$ and so $|e_1 \cap e_2| \geq 2$. Hence $\mathcal{H}_k(R)$ is not linear. $\square$

Theorem 2.6. *Let $R = F_1 \times \cdots \times F_n$ be a finite commutative ring, where each F_i is a field and $n \geq 3$. Then $\mathcal{H}_n(R)$ is regular if and only if $F_i \cong F_j$ for all i, j, $1 \leq i, j \leq n$.*

Proof. Assume that $\mathcal{H}_n(R)$ is regular. By Remark 2.1, $Z(R, n) = \mathbb{A}_1$. Let $W_\ell = F_1^* \times \cdots \times F_{\ell-1}^* \times \{0\} \times F_{\ell+1}^* \times \cdots \times F_n^*$ for $1 \leq \ell \leq n$. Then $Z(R, n) = \bigcup_{\ell=1}^n W_\ell$. For each $\ell = 1, \dots, n$, and any $x, y \in W_\ell$, $x, y \notin e$ for all $e \in E(\mathcal{H}_n(R))$. Hence every edge in $\mathcal{H}_n(R)$ contains exactly one element from W_ℓ, for $1 \leq \ell \leq n$.

Suppose $F_i \ncong F_j$ for some $i \neq j$. Then $|F_i| < |F_j|$, $d_{\mathcal{H}_k(R)}(v) = \prod_{t=1, t\neq j}^n |W_t|$ for all $v \in W_j$, $d_{\mathcal{H}_k(R)}(u) = \prod_{t=1, t\neq i}^n |W_t|$ for all $u \in W_i$ and so $d_{\mathcal{H}_k(R)}(v) \neq d_{\mathcal{H}_k(R)}(u)$ for all $u \in W_i$ and $v \in W_j$, a contradiction. Hence each F_i 's is isomorphic.

Conversely, suppose each F_i 's is isomorphic. Then $|F_i| = |F_j| = q$ for all $i \neq j$, $d_{\mathcal{H}_k(R)}(v) = (q - 1)^{n-1}$ for all $v \in V(\mathcal{H}_n(R))$ and hence $\mathcal{H}_k(R)$ is regular. $\square$

3. Planarity of $\mathcal{H}_k(R)$

In this section, we characterize all finite commutative rings R with identity whose 3-zero-divisor hypergraph $\mathcal{H}_3(R)$ is planar.

Theorem 3.1. *Let $R = F_1 \times \cdots \times F_n$, where each F_i is a field and $3 \le k \le n$. Then the following are true:*
(i) *For any k, $3 < k < n$, $\mathcal{H}_k(R)$ is non-planar*
(ii) *$\mathcal{H}_3(R)$ is planar if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$*
(iii) *If $n \ne 3$, then $\mathcal{H}_n(R)$ is planar if and only if $R \cong \mathbb{Z}_2^n$.*

Proof. (i) Let $x_1 = (\underbrace{0, 1, \ldots, 1}_{k\text{-terms}}, \overbrace{0, \ldots, 0}^{n-k\ terms})$, $x_2 = (\underbrace{1, 0, 1, \ldots, 1}_{k\text{-terms}}, \overbrace{0, \ldots, 0}^{n-k\ terms})$, $\cdots$, $x_k = (\underbrace{1, 1, \ldots, 1, 0}_{k\text{-terms}}, \overbrace{0, \ldots, 0}^{n-k\ terms})$, $x_2' = (\underbrace{1, 0, 1, \ldots, 1}_{k\text{-terms}}, \overbrace{1, 0, \ldots, 0}^{n-k\ terms})$, $x_3' = (\underbrace{1, 1, 0, 1, \ldots, 1}_{k\text{-terms}}, \overbrace{1, 0, \ldots, 0}^{n-k\ terms}) \in V(\mathcal{H}_k(R))$. Then $e_1 = \{x_1, \ldots, x_k\}$, $e_2 = \{x_1, x_2', x_3, \ldots, x_k\}$, $e_3 = \{x_1, x_2, x_3', x_4, \ldots, x_k\} \in E(\mathcal{H}_k(R))$.

Consider $\Omega_1 = \{x_1, \ldots, x_k, x_2', x_3', e_1, e_2, e_3\}$. Then the subgraph induced by Ω_1 of $\mathcal{I}(\mathcal{H}_k(R))$ contains a subdivision of $K_{3,3}$. By Theorems 1.1 and 1.2, $\mathcal{H}_k(R)$ is non-planar.

(ii) Assume that $\mathcal{H}_3(R)$ is planar. Suppose $n > 3$. For $a \in R_1^*, b \in R_2^*, c \in R_3^*, d \in R_4^*$, let $x_1 = (a, b, 0, \ldots, 0)$, $x_2 = (a, 0, c, 0, 0, \ldots, 0)$, $x_3 = (0, b, c, 0, \ldots, 0)$, $x_4 = (a, b, c, 0, \ldots, 0)$, $x_5 = (a, 0, 0, d, 0, \ldots, 0)$, $x_6 = (0, b, 0, d, 0, \ldots, 0)$, $x_7 = (a, b, 0, d, 0, \ldots, 0)$, $x_8 = (0, 0, c, d, 0, \ldots, 0)$, $x_9 = (a, 0, c, d, 0, \ldots, 0)$, $x_{10} = (0, b, c, d, 0, \ldots, 0) \in V(\mathcal{H}_k(R))$.

Then $e_1 = \{x_1, x_2, x_3\}$, $e_2 = \{x_7, x_2, x_3\}$, $e_3 = \{x_7, x_2, x_8\}$, $e_4 = \{x_6, x_8, x_3\}$, $e_5 = \{x_1, x_6, x_9\}$, $e_6 = \{x_1, x_2, x_{10}\}$, $e_7 = \{x_5, x_2, x_{10}\}$, $e_8 = \{x_4, x_5, x_{10}\}$, $e_9 = \{x_4, x_6, x_9\}$ and $e_{10} = \{x_6, x_9, x_3\}$ are edges in $E(\mathcal{H}_3(R))$. Now the subgraph induced by $\Omega_2 = \{x_1, \ldots, x_{10}, e_1, \ldots, e_{10}\}$ of $\mathcal{I}(\mathcal{H}_3(R))$ contains a subdivision of $K_{3,3}$ and by Theorems 1.1 and 1.2, $\mathcal{H}_3(R)$ is non-planar, a contradiction. Hence $n = 3$ and so $R = R_1 \times R_2 \times R_3$.

Suppose for some i, $|R_i| \ge 4$. Without loss of generality, we assume that $|R_3| \ge 4$. Let $a_1, a_2, a_3 \in R_3^*$. Then $y_1 = (a, b, 0)$, $y_2 = (0, b, a_1)$, $y_3 = (a, 0, a_1)$, $y_4 = (0, b, a_2)$, $y_5 = (a, 0, a_1)$, $y_6 = (0, b, a_3)$, $y_7 = (a, 0, a_3) \in V(\mathcal{H}_k(R))$, where $a \in R_1^*, b \in R_2^*$. Let $f_1 = \{y_1, y_2, y_3\}$, $f_2 = \{y_1, y_4, y_3\}$, $f_3 = \{y_1, y_2, y_5\}$, $f_4 = \{y_1, y_4, y_5\}$, $f_5 = \{y_1, y_6, y_3\}$, $f_6 = \{y_1, y_2, y_7\}$, $f_7 = \{y_1, y_7, y_6\} \in E(\mathcal{H}_k(R))$. Let $\Omega_2 = \{y_1, \ldots, y_7, f_1, \ldots, f_7\}$. Then the subgraph induced by Ω_2 of $\mathcal{I}(\mathcal{H}_3(R))$ contains a subdivision of $K_{3,3}$ and by Theorems 1.1 and 1.2, $\mathcal{H}_3(R)$ is non-planar, a contradiction. Hence $|R_i| < 4$ for $i = 1, 2, 3$.

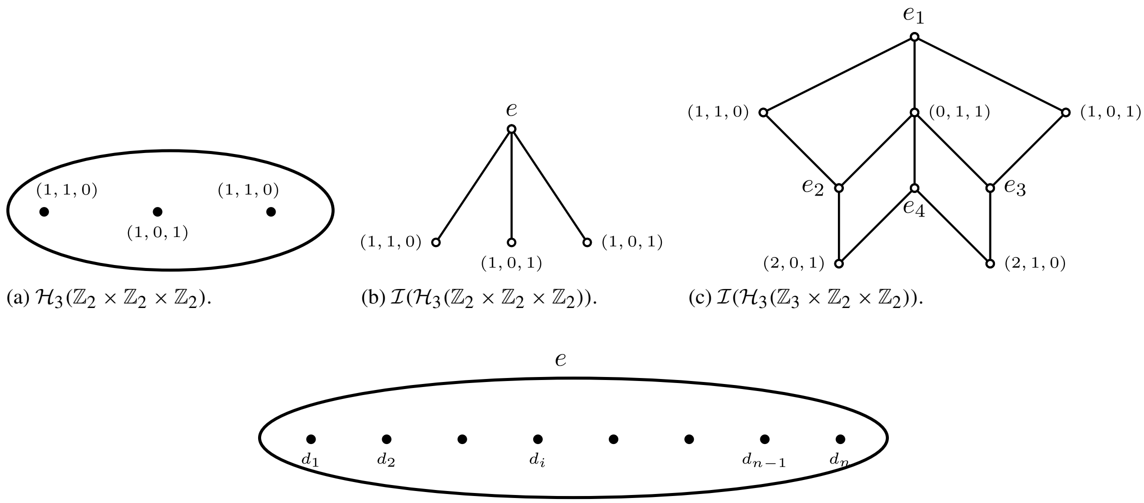
Suppose there exist at least two R_i 's such that $|R_i| = 3$. Without loss of generality, we assume that $|R_1| = |R_2| = 3$. Then $R_1 = R_2 = \mathbb{Z}_3$. Let $z_1 = (1, 2, 0)$, $z_2 = (1, 1, 0)$, $z_3 = (2, 2, 0)$, $z_4 = (2, 1, 0)$, $z_5 = (0, 2, c)$, $z_6 = (0, 1, c)$, $z_7 = (1, 0, c)$ and $z_8 = (2, 0, c) \in V(\mathcal{H}_k(R))$, where $c \in R_3^*$. Then $g_1 = \{z_1, z_5, z_7\}$, $g_2 = \{z_3, z_5, z_7\}$, $g_3 = \{z_1, z_6, z_8\}$, $g_4 = \{z_3, z_6, z_8\}$, $g_5 = \{z_1, z_6, z_7\}$, $g_6 = \{z_4, z_5, z_8\}$ and $g_7 = \{z_4, z_6, z_8\} \in E(\mathcal{H}_k(R))$. Consider $\Omega_3 = \{z_1, \ldots, z_8, g_1, \ldots, g_7\}$. Then the incidence subgraph induced by Ω_3 of $\mathcal{I}(\mathcal{H}_3(R))$ contains a subdivision of $K_{3,3}$ and by Theorems 1.1 and 1.2, $\mathcal{H}_3(R)$ is non-planar, a contradiction. Hence $R \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and the corresponding planar embedding of these commutative rings is given in Fig. 2.1(b) and (c).

Converse is obvious.

(iii) Assume that $\mathcal{H}_n(R)$ is planar. Suppose $|R_i| \ge 3$ for some i. Without loss of generality, we assume that $|R_1| \ge 3$. Let $d_i = (1, \ldots, 1, 0, 1, \ldots, 1) \in V(\mathcal{H}_n(R))$ where 0 is in the ith place of d_i, $1 \le i \le n$. For $a \ne 1$, let $c_2 = (a, 0, 1, \ldots, 1)$, $c_3 = (a, 1, 0, 1, \ldots, 1) \in V(\mathcal{H}_n(R))$. Then $e_1 = \{d_1, d_2, \ldots, d_n\}$, $e_2 = \{d_1, c_2, d_3, \ldots, d_n\}$, $e_3 = \{d_1, d_2, c_3, d_4, \ldots, d_n\} \in E(\mathcal{H}_n(R))$.

Let $\Omega_4 = \{d_1, \ldots, d_n, c_2, c_3, e_1, e_2, e_3\}$. Then the subgraph induced by Ω_4 of $\mathcal{I}(\mathcal{H}_n(R))$ contains a subdivision of $K_{3,3}$ and by Theorems 1.1 and 1.2, $\mathcal{H}_n(R)$ is non planar, a contradiction. Hence $|R_i| = 2$ for all i, $1 \le i \le n$ and so $R \cong \mathbb{Z}_2^n$. A planar embedding of $\mathbb{Z}_2^n$ is given in Fig. 2.1(d). Converse is obvious. □

Theorem 3.2. *Let $R = R_1 \times \cdots \times R_n$ be a finite commutative ring, where each $(R_i, \mathfrak{m}_i)$ is local ring but not a field and $n \ge 2$. Then $\mathcal{H}_k(R)$ is non-planar for all k, $3 \le k \le n$.*



(a) $\mathcal{H}_3(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$. (b) $\mathcal{I}(\mathcal{H}_3(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))$. (c) $\mathcal{I}(\mathcal{H}_3(\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2))$.

(d) $\mathcal{H}_n(\mathbb{Z}_2^n)$.

Fig. 2.1. Planar Embeddings.

Proof. Suppose $k = 3$. Since R_1 and R_2 are local rings, there exist $a \in \mathfrak{m}_1^*$ and $b \in \mathfrak{m}_2^*$ such that $ann(a) = \mathfrak{m}_1$ and $ann(b) = \mathfrak{m}_2$. Note that $|R_1^\times| \geq 2$ and $|R_2^\times| \geq 2$. Let $u_1, u_2 \in R_1^\times$ and $v_1, v_2 \in R_2^\times$. Let $x_1 = (0, v_1, 0, \dots, 0)$, $x_2 = (u_1, 0, \dots, 0)$, $x_3 = (u_1, b, 0, \dots, 0)$, $x_4 = (a, v_1, 0, \dots, 0)$, $x_5 = (a, b, 0, \dots, 0)$, $x_6 = (a, v_2, 0, \dots, 0)$, $x_7 = (u_2, b, 0, \dots, 0) \in Z(R, k)$. Then $e_1 = \{x_3, x_4, x_5\}$, $e_2 = \{x_3, x_6, x_5\}$, $e_3 = \{x_2, x_4, x_6\}$, $e_4 = \{x_7, x_4, x_5\}$, $e_5 = \{x_3, x_1, x_7\} \in E(\mathcal{H}_k(R))$. Let $\Omega_1 = \{x_1, \dots, x_7, e_1, \dots, e_5\}$. Then the subgraph induced by Ω_1 of $\mathcal{I}(\mathcal{H}_k(R))$ contains a subdivision of $K_{3,3}$ and by Theorems 1.1 and 1.2, $\mathcal{H}_3(R)$ is non-planar.

Suppose $3 < k \leq n$. Note that $\mathcal{H}_k(\mathbb{Z}_2^n)$ is a subgraph of $\mathcal{H}_k(R)$. By Theorem 3.1, $\mathcal{H}_k(R)$ is non-planar. □

Theorem 3.3. *$R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring, $m \geq 1$, $n \geq 1$ where each $(R_i, \mathfrak{m}_i)$ is a local ring but not a field and each F_j is a field. Then $\mathcal{H}_3(R)$ is planar if and only if R is isomorphic to one of the following rings:*

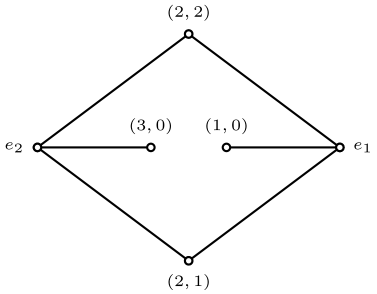
$$\mathbb{Z}_4 \times \mathbb{Z}_3,\ \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_3,\ \mathbb{Z}_4 \times \mathbb{F}_4,\ \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{F}_4,\ \mathbb{Z}_9 \times \mathbb{Z}_2,\ \frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2.$$

Proof. Note that $Z(\mathbb{Z}_4 \times \mathbb{Z}_2, 3)^* = Z\left(\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2, 3\right)^* = \emptyset$. Then $R \ncong \mathbb{Z}_4 \times \mathbb{Z}_2$ and $R \ncong \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$.

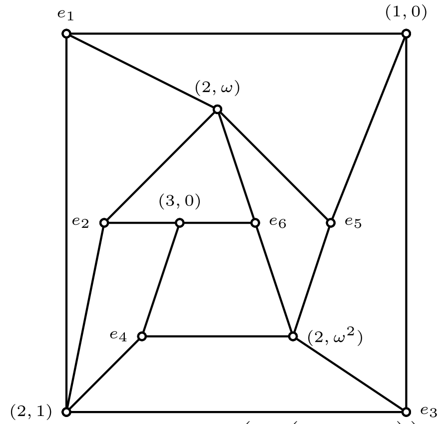
Assume that $\mathcal{H}_k(R)$ is planar. Suppose $n \geq 2$. Then by Theorem 3.2, $\mathcal{H}_k(R)$ is non-planar. Hence $n = 1$. Since R_1 is not a field, there exists $a \in \mathfrak{m}_1^*$ such that $a^2 = 0$.

Suppose $m \geq 2$. Let $x_1 = (0, 0, 1, 0, \dots, 0)$, $x_2 = (0, 1, 0, \dots, 0)$, $x_3 = (0, 1, 1, 0, \dots, 0)$, $x_4 = (a, 0, 0, \dots, 0)$, $x_5 = (a, 0, 1, 0, \dots, 0)$, $x_6 = (a, 1, 0, \dots, 0)$, $x_7 = (a, 1, 1, 0, \dots, 0)$, $x_8 = (1, 0, 0, \dots, 0)$, $x_9 = (1, 0, 1, 0, \dots, 0)$, $x_{10} = (1, 1, 0, \dots, 0)$, $x_{11} = (u_1, 0, 0, \dots, 0)$, $x_{12} = (u_1, 0, 1, 0, \dots, 0)$, $x_{13} = (u_1, 1, 0, \dots, 0) \in Z(R, 3)^*$, where $1 \neq u_1$ is unit in R_1. Then $e_1 = \{x_3, x_9, x_{10}\}$, $e_2 = \{x_3, x_6, x_9\}$, $e_3 = \{x_3, x_9, x_{13}\}$, $e_4 = \{x_3, x_5, x_{10}\}$, $e_5 = \{x_3, x_5, x_{13}\}$, $e_6 = \{x_3, x_{10}, x_{12}\}$, $e_7 = \{x_3, x_6, x_{12}\}$, $e_8 = \{x_3, x_{13}, x_{12}\}$, $e_9 = \{x_6, x_7, x_9\}$, $e_{10} = \{x_7, x_6, x_{12}\}$, $e_{11} = \{x_5, x_7, x_{10}\}$, $e_{12} = \{x_5, x_7, x_{13}\} \in E(\mathcal{H}_3(R))$. Consider $\Omega = \{x_1, \dots, x_{13}, e_1, \dots, e_{12}\}$. Then the subgraph induced by Ω of $\mathcal{I}(\mathcal{H}_3(R))$ contains a subdivision of $K_{3,3}$ and so $\mathcal{H}_3(R)$ is non-planar. Hence $m = 1$ and so $R \cong R_1 \times F_1$.

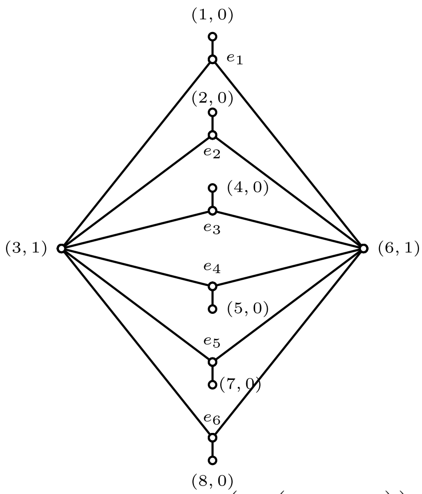
Suppose $|F_1| \geq 5$. Let $y_1 = (a, 1)$, $y_2 = (a, v_1)$, $y_3 = (u_1, 0)$, $y_4 = (u_2, 0)$, $y_5 = (a, v_2)$, $y_6 = (a, v_3) \in Z(R, 3)^*$, where v_1, v_2, and v_3 are distinct nonzero non-identity element in F_1^*. Then $f_1 = \{y_1, y_2, y_3\}$, $f_2 = \{y_1, y_2, y_4\}$, $f_3 = \{y_2, y_3, x_5\}$, $f_4 = \{y_2, y_4, y_5\}$, $f_5 = \{y_2, y_3, y_6\}$, $f_6 = \{y_1, y_3, x_6\} \in E(\mathcal{H}_3(R))$. Consider $\Omega_1 = \{y_1, \dots, y_6, f_1, \dots, f_6\}$. Then the subgraph induced by Ω_1 of $\mathcal{I}(\mathcal{H}_3(R))$ contains a subdivision of $K_{3,3}$ and so $\mathcal{H}_3(R)$ is non-planar, a contradiction. Hence $|F_1| \leq 4$.



(a) $\mathcal{I}(\mathcal{H}_3(\mathbb{Z}_4 \times \mathbb{Z}_3)) \cong \mathcal{I}\left(\mathcal{H}_3\left(\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{Z}_3\right)\right)$.



(b) $\mathcal{I}(\mathcal{H}_3(\mathbb{Z}_4 \times \mathbb{F}_4)) \cong \mathcal{I}\left(\mathcal{H}_3\left(\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{F}_4\right)\right)$.



(c) $\mathcal{I}(\mathcal{H}_3(\mathbb{Z}_9 \times \mathbb{Z}_2)) \cong \mathcal{I}\left(\mathcal{H}_3\left(\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle} \times \mathbb{Z}_2\right)\right)$.

Fig. 2.2. Planar Embeddings.

Suppose $|\mathfrak{m}_1^*| \geq 3$. Then $|R_1^\times| \geq 4$. Let $b, c \in \mathfrak{m}_1^*$ such that $ab = ac = 0$ and $b \neq a \neq c$. Let $z_1 = (a, 1)$, $z_2 = (b, 1)$, $z_3 = (c, 1)$, $z_4 = (u_1, 0)$, $z_5 = (u_2, 0)$, $z_6 = (u_3, 0)$, $z_7 = (u_4, 0) \in Z(R, 3)^*$, where $u_1, u_2, u_3, u_4 \in R_1^\times$. Then $e_1 = \{z_1, z_2, z_4\}$, $e_2 = \{z_1, z_2, z_5\}$, $e_3 = \{z_1, z_3, z_6\}$, $e_4 = \{z_1, z_2, z_7\}$, $e_5 = \{z_1, z_3, z_4\}$, $e_6 = \{z_1, z_3, z_5\}$, $e_7 = \{z_1, z_3, z_6\}$, $e_8 = \{z_1, z_3, z_7\} \in \mathcal{H}_3(R)$. Consider $\Omega_2 = \{z_1, \ldots, z_7, e_1, \ldots, e_8\}$. Then the subgraph induced by Ω_2 of $\mathcal{I}(\mathcal{H}_3(R))$ contains a subdivision of $K_{3,3}$ and so $\mathcal{H}_3(R)$ is non-planar, a contradiction. Hence $|\mathfrak{m}_1^*| \leq 2$.

Suppose $|\mathfrak{m}_1^*| = 1$. Then $R_1 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$.

Since $Z(\mathbb{Z}_4 \times \mathbb{Z}_2, 3)^* = Z\left(\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{Z}_2, 3\right)^* = \emptyset$, $F_1 \ncong \mathbb{Z}_2$. Since $|F_1| \leq 4$, $F_1 \cong \mathbb{Z}_3$ or $\mathbb{F}_4$ and hence R is isomorphic to one of the following rings:

$$\mathbb{Z}_4 \times \mathbb{F}_4,\ \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{F}_4,\ \mathbb{Z}_4 \times \mathbb{Z}_3 \text{ or } \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{Z}_3.$$

Suppose $|\mathfrak{m}_1^*| = 2$. Then $R \cong \mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$. If $|F_1| \geq 3$, then $\mathcal{I}(\mathcal{H}_3(R))$ contains $K_{3,3}$ as a subgraph and so $\mathcal{H}_3(R)$ is non-planar, a contradiction. Hence $F_1 \cong \mathbb{Z}_2$. Hence R is isomorphic to one of the following rings:

$$\mathbb{Z}_9 \times \mathbb{Z}_2 \quad \text{or} \quad \frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2.$$

Converse follows from Fig. 2.2(a)–(c). □

Acknowledgments

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